Journal of Algebra 323 (2010) 336–348

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the Brauer group complex for a multiquadratic field extension

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ARTICLE INFO

Article history:
Received 8 April 2009
Available online 10 September 2009
Communicated by Eva Bayer-Fluckiger

Keywords:
Quadratic form
Pfister form
Transfer
Second residue map

ABSTRACT

Let F be a field of characteristic not 2, $a_1,\dots,a_n \in F^*$, $M = F(\sqrt{a_1},\dots,\sqrt{a_n})$. Let further G be the subgroup of F^*/F^{*2} generated by $a_1,\dots,a_n$. For any $I \subset \{1,\dots,n\}$ put $a_I = \prod_{i\in I} a_i$ ($a_\emptyset = 1$). Consider the complex

$$\prod_{I\neq\emptyset} K_I^*/K_I^{*2} \xrightarrow{\varphi} G \otimes F^*/F^{*2} \xrightarrow{\psi} {}_2\mathrm{Br}\,F \xrightarrow{res} {}_2\mathrm{Br}\,M \xrightarrow{\prod N_{M/E}} \prod_{F\subset E\subset M,\,[M:E]=2} {}_2\mathrm{Br}\,E,$$

where $K_I = F(\sqrt{a_I})$ are all the quadratic extensions of F, containing in M, $\varphi(\{w_I\}) = \sum_I a_I \otimes N_{K_I/F}(w_I)$, $\psi(a_I \otimes z) = (a_I, z)$. The homology group of this complex at $i+1$-th term from the left is denoted by $h_i(M/F)$. Independently of $\mathrm{char}\,F$ we give examples of these complexes with nontrivial $h_3(M/F)$ and prescribed $n \geqslant 3$ and $a_1,\dots,a_{n-1}$. Also similar examples concerning the group $h_1(M/F)$ are constructed.

Let F be a field of characteristic not 2, $a_1,\dots,a_n \in F^*$, $M = F(\sqrt{a_1},\dots,\sqrt{a_n})$. Let further G be the subgroup of F^*/F^{*2} generated by $a_1,\dots,a_n$. For any $I \subset \{1,\dots,n\}$ put $a_I = \prod_{i\in I} a_i$ ($a_\emptyset = 1$). Consider the complex

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.07.036

where $K_I = F(\sqrt{a_I})$ are all quadratic extensions of F, containing in M, $\varphi(\{w_I\}) = \sum_I a_I \otimes N_{K_I/F}(w_I)$, $\psi(a_I \otimes z) = (a_I, z)$. We call this complex the Brauer group complex for the given multiquadratic extension M/F (in fact this is the Brauer group complex from [ELTW]). The homology group of this complex at $i+1$-th term from the left is denoted by $h_i(M/F)$. It is known that if $n \leqslant 2$, or F is either local or global field, then $h_i(M/F) = 0$ [T,ELW,ELTW,STW]. A natural question arises whether this complex is always exact, and if not, how big the groups $h_i(M/F)$ can be. The group $h_2(M/F)$ was considered in [T,ELTW,S3]. In particular, numerous examples of nontrivial $h_2(M/F)$ were given in these papers. In the present article we investigate the groups $h_3(M/F)$, and $h_1(M/F)$, devoting a separate section to each of them.

Our notation is almost everywhere as in the books [L] and [Sch]. The only difference is in the definition of Pfister forms; for elements $a_1, \dots, a_n \in F^*$ we put $\langle\langle a_1, \dots, a_n \rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ (take notice of the signs). By ${}_2\mathrm{Br}\, F$ we denote the 2-torsion of the Brauer group of the field F, which is isomorphic to the cohomology group $H^2(F, \mathbb{Z}/2\mathbb{Z})$. If $a, b \in F^*$, then, as usual, $(a, b) \in {}_2\mathrm{Br}\, F$ denotes the class of the quaternion algebra over F, determined by the generators i and j, and the relations $i^2 = a$, $j^2 = b$, $ij = -ji$. Slightly abusing terminology we will often identify a central simple algebra with its class in the Brauer group. If $\alpha \in {}_2\mathrm{Br}\, F$ and L/F is a field extension, then by definition $\alpha_L = \mathrm{res}_{L/F}\, \alpha$, and $\mathrm{res}_{L/F}\, {}_2\mathrm{Br}\, F$ is the image of ${}_2\mathrm{Br}\, F$ in ${}_2\mathrm{Br}\, L$ under the restriction map. Recall also that if the extension L/F is finite, there exists a norm map $\mathrm{N}_{L/F} : {}_2\mathrm{Br}\, L \to {}_2\mathrm{Br}\, F$, which makes the diagram

$$\begin{array}{ccc} {}_2\mathrm{Br}\, L & \longrightarrow & H^2(L, \mathbb{Z}/2\mathbb{Z}) \\ {\scriptstyle \mathrm{N}_{L/F}}\big\downarrow & & {\scriptstyle \mathrm{N}_{L/F}}\big\downarrow \\ {}_2\mathrm{Br}\, F & \longrightarrow & H^2(F, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

commutative, and which is compatible with the restriction. If $a \in F^*$ and $b \in L^*$, then $\mathrm{N}_{L/F}(a, b) = (a, \mathrm{N}_{L/F}\, b)$. The composition $\mathrm{N}_{L/F} \circ \mathrm{res}_{L/F}$ is just the multiplication by the degree of the extension L/F. If L/F is quadratic, then the sequence

$${}_2\mathrm{Br}\, F \xrightarrow{res} {}_2\mathrm{Br}\, L \xrightarrow{\mathrm{N}_{L/F}} {}_2\mathrm{Br}\, F$$

is exact.

We denote by $K_2(F)$ the 2-dimensional Milnor K-group of F. There is an isomorphism $K_2(F)/2K_2(F) \simeq {}_2\mathrm{Br}\, F$ taking the symbol $\{a, b\}$ to (a, b), which is compatible with the restriction and the norm [M1].

1. The group $h_3(M/F)$

We are interested in the group $h_3(M/F)$ partly because the sequence

$$F^*/F^{*2} \xrightarrow{res} M^*/M^{*2} \xrightarrow{\prod N_{M/E}} \prod_{F \subset E \subset M,\, [M:E]=2} E^*/E^{*2}$$

is exact (see [ELW], Theorem 2.1), which means that "one-dimensional" version of $h_3(M/F)$ is zero.

In this section we give examples of multiquadratic extensions M/F such that $h_3(M/F)$ is infinite and $\mathrm{cd}_2\, F = 2$, following an idea in [S1] and [S2].

In order to construct a multiquadratic extension M/F with nonzero $h_3(M/F)$ we start from any field k, $n \geqslant 2$, and elements $a_1, \dots, a_{n-1}, d, u \in k^*$. Put

$$F = k(x), \qquad a_n = x^2 - 4d, \qquad M = F(\sqrt{a_1}, \dots, \sqrt{a_n}),$$

where x is an indeterminate.

Consider the element $\beta=(u,\frac{1}{2}(x+\sqrt{x^2-4d}))\in {}_2\mathrm{Br}\,F(\sqrt{x^2-4d})$. Notice that $N_{F(\sqrt{x^2-4d})/F}\beta=(u,d)$. Our main purpose is to prove the following:

Proposition 1. *The following two conditions are equivalent*:

1) *At least one of the algebras $(a_I u,d)$, where I runs over all the subsets of $\{1,\dots,n-1\}$, is zero.*
2) $\beta_M\in \mathrm{res}_{M/F}\,{}_2\mathrm{Br}\,F$.

Proof. 1) $\Longrightarrow$ 2). Assume that $(a_I u,d)=0$ for some $I\subset\{1,\dots,n-1\}$. Then we have

$$\mathrm{N}_{F(\sqrt{x^2-4d})/F}\left(\beta+\left(a_I,\frac{1}{2}\bigl(x+\sqrt{x^2-4d}\bigr)\right)\right)=(u,d)+(a_I,d)=(a_I u,d)=0.$$

Since $h_3(F(\sqrt{x^2-4d})/F)=0$, we get that

$$\left(\beta+\left(a_I,\frac{1}{2}\bigl(x+\sqrt{x^2-4d}\bigr)\right)\right)\in \mathrm{res}_{F(\sqrt{x^2-4d})/F}\,{}_2\mathrm{Br}\,F.$$

It remains to notice that $\beta_M=\beta_M+(a_I,\frac{1}{2}(x+\sqrt{x^2-4d}))_M\in\mathrm{res}_{M/F}\,{}_2\mathrm{Br}\,F$.

2) $\Longrightarrow$ 1). The proof of this implication is much harder than the proof of the previous one. It suffices to consider the case where k has no proper finite odd degree extensions. Indeed, we can pass to a maximal odd degree extension k'/k and use that the map ${}_2\mathrm{Br}\,k\to{}_2\mathrm{Br}\,k'$ is injective.

Now assume that all the algebras $(a_I u,d)$ are nonzero, and $\beta_M=\alpha_M$ for some $\alpha\in{}_2\mathrm{Br}\,F$. For any field K let

$${}_2\mathrm{Br}\,K(x)\xrightarrow{\partial=\oplus\partial_p}\coprod_{p\in\mathbb{P}^1_K}K(p)^*/K(p)^{*2}$$

be the second residue map. Since any element of ${}_2\mathrm{Br}\,F$ is a sum of quaternion algebras [M1], and $((x^2-4d)a,b)=(x^2-4d,b)+(a,b)$ for any $a,b\in F^*$, we can assume that $\partial_{x^2-4d}(\alpha)=1$. Put $l=k(\sqrt{a_1},\dots,\sqrt{a_{n-1}})$. Notice that since $(a_I u,d)\neq 0$ for any I, the algebra $l(\sqrt{u})=l[x]/(x^2-u)$ is a field. Since $\alpha_{M(\sqrt{u})}=\beta_{M(\sqrt{u})}=0$, we have

$$\alpha_{l(\sqrt{u})(x)}=\bigl(x^2-4d,h\bigr),$$

where $h\in l(\sqrt{u})[x]$. Assume also that $\deg h$ is minimal. In particular, h is a squarefree polynomial.

Lemma 2. *One can assume that $h=ef$, where $f\in k[x]$ is monic, and $e\in l(\sqrt{u})^*$.*

Proof. If $f\in l(\sqrt{u})[x]$ (resp. $f\in k[x]$), then by $\overline{f}$ we will denote the residue of f in the algebra $l(\sqrt{u},\sqrt{d})=l(\sqrt{u})[x]/(x^2-4d)$ (resp. in the field $k(\sqrt{d})=k[x]/(x^2-4d)$).

Obviously,

$$\bigl(x^2-4d,h\bigr)=\alpha_{l(\sqrt{u})(x)}=\bigl(x^2-4d,\sigma h\bigr)$$

for any $\sigma\in Gal(l(\sqrt{u})(x)/F)$. Suppose $p\in l(\sqrt{u})[x]$ is a prime divisor of h, but not a divisor of x^2-4d. Let us show that σp is a prime divisor of h as well. Indeed, assume the contrary. Then

$$1=\partial_{\sigma p}\bigl(x^2-4d,h\bigr)=\partial_{\sigma p}\bigl(x^2-4d,\sigma h\bigr)=x^2-4d\ (\mathrm{mod}\ \sigma p)=\partial_{\sigma p}\bigl(x^2-4d,\sigma p\bigr).$$

Applying the complex

$$ {}_2\mathrm{Br}\, l(\sqrt{u})(x) \xrightarrow{\partial} \coprod_{p\in\mathbb{P}^1_{l(\sqrt{u})}} l(\sqrt{u})(p)^*/l(\sqrt{u})(p)^{*2} \xrightarrow{\mathrm{N}} l(\sqrt{u})^*/l(\sqrt{u})^{*2} $$

(the reciprocity law) to the $l(\sqrt{u})(x)$-algebra $(x^2-4d,\sigma p)$ we get $N_{l(\sqrt{u},\sqrt{d})/l(\sqrt{u})}\overline{\sigma p} \in l(\sqrt{u})^{*2}$. Therefore, $\overline{\sigma p} \in vl(\sqrt{u},\sqrt{d})^{*2}$ for some $v\in l(\sqrt{u})^*$. Comparing the residues, we conclude that $(x^2-4d,h)=(x^2-4d, v\frac{\sigma h}{\sigma p})$, a contradiction to minimality of $\deg h$.

Thus, we have either

1) $h=ef$, or 2) $h=e(x\pm 2\sqrt{d})f)$, where $f\in k[x]$ is monic, and $e\in l(\sqrt{u})^*$. Since $\deg h$ is minimal, and $(x^2-4d,-(x^2-4d))=0$, we see that x^2-4d does not divide f. Of course, case 2) is impossible if $\sqrt{d}\notin l(\sqrt{u})$. We are going to show that in fact case 2) is always impossible, which will complete the proof of the lemma. Indeed, assume that $\sqrt{d}\in l(\sqrt{u})$, and, for instance, $h=(x-2\sqrt{d})P$, where $P\in l(\sqrt{u})[x]$. Then we have

$$ \begin{aligned} \bigl(x^2-4d,(x-2\sqrt{d})P\bigr) &= \bigl(x^2-4d,P\bigr)+\bigl(x^2-4d,x-2\sqrt{d}\bigr) \\ &= \bigl(x^2-4d,P\bigr)+(-x-2\sqrt{d},x-2\sqrt{d})=\bigl(x^2-4d,P\bigr)+\bigl(x^2-4d,-4\sqrt{d}\bigr) \\ &= \bigl(x^2-4d,-4\sqrt{d}P\bigr) \end{aligned} $$

a contradiction since the $\deg P=\deg h-1$, and $\deg h$ is minimal. □

Lemma 3. *In the notation of Lemma* 2 *one can assume that* $e\in l^*$.

Proof. Since by our assumption $\partial_{x^2-4d}(\alpha)=1$, we have

$$ 1=\partial_{x^2-4d}\alpha_{l(\sqrt{u})(x)}=\overline{ef}, $$

i.e. $\overline{f}\in el(\sqrt{u},\sqrt{d})^{*2}$, which implies, taking the norm, that $N_{l(\sqrt{u},\sqrt{d})/l(\sqrt{u})}\overline{f}\in l(\sqrt{u})^{*2}$. Since the norm map commutes with the restriction, and $f\in k[x]$, we get that either $N_{k(\sqrt{d})/k}\overline{f}\in a_I k^{*2}$, or $N_{k(\sqrt{d})/k}\overline{f}\in a_I uk^{*2}$ for some $I\subset\{1,\dots,n-1\}$. Since by the hypothesis all the algebras $(a_I u,d)$ are nontrivial, and the composition

$$ k(\sqrt{d})^*/k(\sqrt{d})^{*2} \xrightarrow{\mathrm{N}} k^*/k^{*2} \xrightarrow{\cup d} {}_2\mathrm{Br}\, k $$

is zero, we conclude that $N_{k(\sqrt{d})/k}\overline{f}\in a_I k^{*2}$, or, equivalently, $N_{l(\sqrt{d})/l}\overline{f}\in l^{*2}$. Therefore, $\overline{f}\in e_0 l(\sqrt{d})^{*2}$ for some $e_0\in l^*$. Comparing the residues, we get

$$ \bigl(x^2-4d,ef\bigr)=\bigl(x^2-4d,e_0 f\bigr)_{l(\sqrt{u})(x)}. $$

Hence, changing e to e_0, we can assume that $e\in l^*$. □

Thus, since $(\alpha-(x^2-4d,ef))_{l(\sqrt{u})(x)}=0$, we obtain

$$ \bigl(\alpha-\bigl(x^2-4d,ef\bigr)\bigr)_{l(x)}=(u,g), \tag{$*$} $$

where $e \in l^*$ and $g \in l[x]$. It follows from $(*)$ that for any $\sigma \in Gal(l/k)$,

$$(u, g) = (u, \sigma g) + \big(x^2 - 4d, e\sigma e\big).$$

Hence for any prime $p \in l[x]$ dividing g, but not dividing $x^2 - 4d$ we have $\partial_{\sigma p}(u, g) = \partial_{\sigma p}(u, \sigma g)$. Repeating the proof of Lemma 2 we conclude that either

1) $g = \widetilde{e}\widetilde{f}$, or
2) $g = \widetilde{e}(x \pm 2\sqrt{d})\widetilde{f}$,

where $\widetilde{e} \in l^*$, and $\widetilde{f} \in k[x]$.

Suppose that case 2) holds. This is possible only if $d \in a_I k^{*2}$ for some $I \subset \{1, \dots, n-1\}$. Consider any field k' such that $k(\sqrt{a_I}) \subset k' \subset l$, and $[l : k'] = 2$. Applying the norm $N_{l(x)/k'(x)}$ to the equality $(*)$ we get

$$\big(x^2 - 4d, N_{l(x)/k'(x)}e\big) = (u, N_{l(x)/k'(x)}\widetilde{e}).$$

Since the polynomial $x^2 - 4d$ is monic, we conclude, taking the specialization of the last equality at the infinity point, that $(u, N_{l(x)/k'(x)}\widetilde{e}) = 0$. Since $x^2 - 4d$ decomposes into linear factors over k', it follows that $N_{l(x)/k'(x)}e \in k'^{*2}$. Taking into account that k' is an arbitrary field of codimension 2 between $k(\sqrt{a_I})$ and l, we can assume by [ELW], Theorem 2.1, that $e \in k(\sqrt{a_I})$.

Now consider a field $k \subset l_1 \subset l$ such that $l = l_1(\sqrt{a_I})$, and apply the norm $N_{l(x)/l_1(x)}$ to the equality $(*)$. We get

$$\big(x^2 - 4d, uN_{k(\sqrt{a_I})/k}e\big)_{l_1(x)} = (u, N_{l/l_1}\widetilde{e})_{l_1(x)}.$$

As earlier, taking the specialization at infinity, we conclude that both parts of the last equality are zero. Since $\sqrt{d} \in l^*$, it follows that $uN_{k(\sqrt{a_I})/k}e \in l^{*2}$. This implies that $uN_{k(\sqrt{a_I})/k}e \in a_J k^{*2}$ for some $J \subset \{1, \dots, n-1\}$, i.e. $N_{k(\sqrt{a_I})/k}e \in a_J uk^{*2}$. Therefore, $(a_J u, d) = (a_J u, a_I) = 0$, a contradiction, so that case 2) does not hold.

Thus, case 1) holds. Put $t = \frac{1}{2}(x + \sqrt{x^2 - 4d})$. Then

$$\frac{d}{t} = \frac{1}{2}\big(x - \sqrt{x^2 - 4d}\big), \qquad x = t + \frac{d}{t}, \qquad \sqrt{x^2 - 4d} = t - \frac{d}{t}.$$

Obviously, $F = k(x) = k(t + \frac{d}{t})$, $F(\sqrt{x^2 - 4d}) = k(t)$. The equality $(*)$ implies

$$(u, t) = \alpha_{l(t)} = \left(u, \widetilde{e}\widetilde{f}\left(t + \frac{d}{t}\right)\right)_{l(t)}.$$

Since $\partial_t(u, \widetilde{e}\widetilde{f}(t + \frac{d}{t})) = u \neq 1$, we conclude that $\deg \widetilde{f}$ with respect to x is odd. Since k has no proper odd degree extensions, and $\widetilde{f}$ is defined over k, it follows that $\widetilde{f}$ is divisible by some linear polynomial $x - m$, where $m \in k$. Therefore, over l we have

$$1 = \partial_{t^2 - mt + d}(u, t) = \partial_{t^2 - mt + d}\left(u, \widetilde{e}\widetilde{f}\left(t + \frac{d}{t}\right)\right) = u.$$

Since u is not a square, $\frac{m^2 - 4d}{u} \in l^{*2}$. Furthermore, since $m \in k$, we get that $\frac{m^2 - 4d}{u} \in a_I k^{*2}$ for some $I \subset \{1, \dots, n-1\}$, hence $(a_I u, d) = 0$. This is a contradiction, which proves implication 2) $\Longrightarrow$ 1) in Proposition 1. $\square$

Corollary 4. *Assume that the equivalent conditions of Proposition* 1 *do not hold, i.e. all the algebras* $(a_I u, d)$ *are nonzero, and* $n \geqslant 3$. *Suppose also that* a_1 *and* a_2 *determine distinct nontrivial elements in* k^*/k^{*2}, *and*

$$(u,d)_{k(\sqrt{a_1})} = (u,d)_{k(\sqrt{a_2})} = (u,d)_{k(\sqrt{a_1 a_2})} = 0.$$

Then β *is a nonzero element in the homology group* $h_3(M/F)$.

Proof. Notice first that to provide the last condition we can replace the field k from Proposition 1 by the composite of the function fields of the forms $\langle ud, -u, -d, a_1\rangle$, $\langle ud, -u, -d, a_2\rangle$, and $\langle ud, -u, -d, a_1 a_2\rangle$. This procedure does not affect nontriviality of the algebras $(a_I u, d)$.

Consider any field E such that $F \subset E \subset M$, and $[M:E]=2$. If $\sqrt{x^2-4d} \in E$, then $N_{M/E}\beta_M = 0$. If $\sqrt{x^2-4d} \notin E$, then $M = E(\sqrt{x^2-4d})$, and $N_{M/E}\beta_M = (u,d)_E$. If $\sqrt{a_1} \notin E$, and $\sqrt{a_2} \notin E$, then $M = E(\sqrt{a_1}) = E(\sqrt{a_2})$, which implies that $a_1 a_2 \in E^{*2}$, or, equivalently, $F(\sqrt{a_1 a_2}) \subset E$. Thus, independently of the choice of E we have $N_{M/E}\beta_M = 0$, hence by Proposition 1 β is a nonzero element in $h_3(M/F)$. $\square$

Along with the Brauer group complex one can consider the analogous complex of the Witt groups

$$W(F) \xrightarrow{res} W(M) \xrightarrow{\prod N_{M/E}} \prod_{F \subset E \subset M,\, [M:E]=2} W(E),$$

where $\mathrm{N}_{M/E}$ denotes the transfer map corresponding to the E-linear map $s^{M/E} : M = E(\sqrt{e}) \to E$ defined by $s^{M/E}(1) = 0$ and $s^{M/E}(\sqrt{e}) = 1$ [Sch].

Corollary 5. *Under conditions of Corollary* 4 *the element* $\langle\langle u, \frac{1}{2}(x+\sqrt{x^2-4d})\rangle\rangle$ *is nontrivial in the homology group of the Witt group complex for the extension* M/F.

Proof. The equality $\mathrm{N}_{M/E}\langle\langle u, \frac{1}{2}(x+\sqrt{x^2-4d})\rangle\rangle = 0$ follows from the definition of the map $\mathrm{N}_{M/E}$. Suppose now that $\langle\langle u, \frac{1}{2}(x+\sqrt{x^2-4d})\rangle\rangle \in \mathrm{res}_{M/F} W(F)$. It is easy to see that then there is $\widetilde{\alpha} \in I^2(F)$ such that $\langle\langle u, \frac{1}{2}(x+\sqrt{x^2-4d})\rangle\rangle = \mathrm{res}_{M/F}\widetilde{\alpha}$. Hence $\beta_M = \mathrm{res}_{M/F}\alpha$, where α is the image of $\widetilde{\alpha}$ under the map $I^2(F) \to I^2(F)/I^3(F) \simeq {}_2\mathrm{Br}(F)$, a contradiction. $\square$

Remark. Slightly modifying the example in Corollary 4, we can give an example of a field F such that the group $h_3(M/F)$ is infinite. Indeed, assume that A is some set of indices, and instead of one element $u \in k^*$ we have a collection of elements u_i $(i \in A)$ such that $(u_i u_j, d) \neq 0$, $(u_i, d)_{k'} = 0$, for any $i \neq j \in A$ and k' is any of the fields $k(\sqrt{a_1})$, $k(\sqrt{a_2})$, $k(\sqrt{a_1 a_2})$. Then by Proposition 1 the algebras $(u_i, \frac{1}{2}(x+\sqrt{x^2-4d}))$ are pairwise distinct elements in $h_3(M/F)$. Similarly, the 2-fold Pfister forms $\langle\langle u_i, \frac{1}{2}(x+\sqrt{x^2-4d})\rangle\rangle$ determine pairwise distinct elements in the corresponding homology group of the Witt group complex.

Proposition 1 implies the following curious statement.

Corollary 6. *Let* k *be a field,* $n \geqslant 2$, $a_1, \ldots, a_{n-1}, d \in k^*$, $F = k(x)$, $P_i, Q_i \in F$, *and the element* $\theta = \sum_i (a_i, P_i^2 - (x^2-4d)Q_i^2)$ *has no residues. Then* $\theta = (a_I, d)$ *for some* $I \subset \{1, \ldots, n-1\}$. *Moreover, any algebra* (a_I, d) *can be written in this form.*

Proof. Since θ has no residues, $\theta \in {}_2\mathrm{Br}\,k$. The specialization of θ at the point $x = 2\sqrt{d}$ is, obviously, zero. Hence $\theta_{k(\sqrt{d})} = 0$, which implies that $\theta = (u,d)$ for some $u \in k^*$. Therefore,

$$N_{F(\sqrt{x^2-4d})/F}\left(\left(u, \frac{1}{2}\left(x+\sqrt{x^2-4d}\right)\right)+\sum_{i=1}^{n-1}\left(a_i, P_i+Q_i\sqrt{x^2-4d}\right)\right)=(u,d)+(u,d)=0,$$

which means that

$$\left(u, \frac{1}{2}\left(x+\sqrt{x^2-4d}\right)\right)+\sum_{i=1}^{n-1}\left(a_i, P_i+Q_i\sqrt{x^2-4d}\right)\in \operatorname{res}_{F(\sqrt{x^2-4d})/F}\, {}_2\mathrm{Br}\, F.$$

It follows that

$$\left(u, \frac{1}{2}\left(x+\sqrt{x^2-4d}\right)\right)\in \operatorname{res}_{F(\sqrt{a_1},\dots,\sqrt{a_{n-1}},\sqrt{x^2-4d})/F}\, {}_2\mathrm{Br}\, F.$$

By Proposition 1 we get $(a_I u, d)=0$ for some $I\subset\{1,\dots,n-1\}$, i.e. $\theta=(u,d)=(a_I,d)$. Notice also that

$$(a_I,d)=\sum_{i\in I}\left(a_i, x^2-\left(x^2-4d\right)\right),$$

which completes the proof. □

Remark. The fact that the Brauer group complex is exact in the cases of quadratic and biquadratic extensions allows us to give another description of the group $h_3(M/F)$ for a multiquadratic extension M/F. Namely,

$$h_3(M/F)=\frac{\bigcap_{F\subset E\subset M,\,[M:E]=2}\operatorname{res}_{M/E}\, {}_2\mathrm{Br}\, E}{\operatorname{res}_{M/F}\, {}_2\mathrm{Br}\, F}=\frac{\bigcap_{F\subset K\subset M,\,[M:K]=4}\operatorname{res}_{M/K}\, {}_2\mathrm{Br}\, K}{\operatorname{res}_{M/F}\, {}_2\mathrm{Br}\, F}.$$

Indeed, this follows from the fact that the complex

$${}_2\mathrm{Br}\, K \xrightarrow{res} {}_2\mathrm{Br}\, M \xrightarrow{\prod N_{M/E}} \prod_{K\subset E\subset M,\,[M:E]=2} {}_2\mathrm{Br}\, K$$

is exact for any quadratic or biquadratic extension M/K. There is a similar description for the homology group of the Witt group complex.

Returning to the field F in Corollary 4, notice that $\mathrm{cd}_2 F\geqslant 3$. However, it is possible to improve this result, and give an example of a field extension $\widehat{F}/F$ linearly disjoint with M/F such that the natural map $h_3(M/F)\to h_3(M\widehat{F}/\widehat{F})$ is injective and any 5-dimensional form over $\widehat{F}$ is isotropic. In particular, $\mathrm{cd}_2\widehat{F}=2$. Recall that G is the subgroup of F^*/F^{*2} generated by a_i, $1\leqslant i\leqslant n$. For an arbitrary field K and a quadratic form φ over K denote by $K(\varphi)$ the function field of the projective quadric determined by φ.

The key point of the modified construction is the following:

Proposition 7. *Let K be a field, M/K a multiquadratic extension, $E_1\subset E_2$ be field extensions of K linearly disjoint with M/F. Then the map $h_3(ME_1/E_1)\to h_3(ME_2/E_2)$ is injective in the following cases*:

1) *E_2/E_1 is an odd degree extension* (*not necessarily finite*).
2) *E_2/E_1 is a purely transcendental extension* (*for example, $E_2=E_1(\varphi)$, where φ is an isotropic quadratic form over E_1*).

3) *$E_2 = E_1(X)$, where X is the conic associated to a quaternion algebra Q over E_1 such that $Q_{ME_1} \neq 0$, $Q \cup (s) \neq 0$ for any $1 \neq s \in G$.*
4) *$E_2 = E_1(\varphi)$, where φ is a 4-dimensional form over E_1 such that $\operatorname{disc}\varphi \notin G$ and $\langle\langle \operatorname{disc}\varphi, s\rangle\rangle \neq 0$ for any $1 \neq s \in G$.*
5) *$E_2 = E_1(\varphi)$, where φ is any 5-dimensional form over E_1.*

Proof. 1) We can assume that the extension E_2/E_1 is finite. Suppose $\beta \in {}_2\mathrm{Br}\, ME_1$ is such an element that $\mathrm{res}_{ME_2/ME_1}\beta = \mathrm{res}_{ME_2/E_2}\alpha$ for some $\alpha \in {}_2\mathrm{Br}\, E_2$. Consider the commutative diagram

$$\begin{array}{ccc} {}_2\mathrm{Br}\, E_2 & \xrightarrow{\mathrm{res}} & {}_2\mathrm{Br}\, ME_2 \\ {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow \\ {}_2\mathrm{Br}\, E_1 & \xrightarrow{\mathrm{res}} & {}_2\mathrm{Br}\, ME_1 \end{array}$$

Since for a finite odd degree extension the composition $\mathrm{N} \circ \mathrm{res}$ is identity, we have

$$\begin{aligned} \mathrm{res}_{ME_2/ME_1} \circ \mathrm{res}_{ME_1/E_1} \circ \mathrm{N}_{E_2/E_1}\alpha &= \mathrm{res}_{ME_2/ME_1} \circ \mathrm{N}_{ME_2/ME_1} \circ \mathrm{res}_{ME_2/E_2}\alpha \\ &= \mathrm{res}_{ME_2/ME_1} \circ \mathrm{N}_{ME_2/ME_1} \circ \mathrm{res}_{ME_2/ME_1}\beta = \mathrm{res}_{ME_2/ME_1}\beta. \end{aligned}$$

Since $[ME_2 : ME_1]$ is odd, the map res_{ME_2/ME_1} is injective, hence

$$\beta = \mathrm{res}_{ME_1/E_1} \circ \mathrm{N}_{E_2/E_1}\alpha,$$

which proves in particular that the map $h_3(ME_1/E_1) \to h_3(ME_2/E_2)$ is injective.

2) We can assume that $E_2 = E_1(t)$. The proof of this part is quite similar to the proof of part 1); the only difference is that the norm map should be changed to the specialization map ${}_2\mathrm{Br}\, E_1(t) \to {}_2\mathrm{Br}\, E_1$ with respect to an arbitrary rational point of the projective line $\mathbb{P}^1_{E_1}$.

3) Suppose $\beta \in {}_2\mathrm{Br}\, ME_1$ is such an element that $\mathrm{res}_{ME_2/ME_1}\beta = \mathrm{res}_{ME_2/E_2}\alpha$ for some $\alpha \in {}_2\mathrm{Br}\, E_2$. Then for any closed point $p \in X$ $\partial_p(\alpha)_{ME_1(p)} = 1$. Hence $\partial_p(\alpha) = s_p$ for some $s_p \in G$ depending on p. The element s_p is not always uniquely determined, but if $\partial_p(\alpha) = 1$, we put $s_p = 1$. Choose any point $p_0 \in X$ of degree 2. Since any divisor on X of degree 0 is principal, there exists a function $f_p \in E_1(X)^*$ such that $\mathrm{div}(f_p) = \frac{\deg p}{2} p_0 - p$. Then the element $\alpha' = \alpha - \sum_{p \in X}(s_p, f_p)$ has at most one nonzero residue, namely at the point p_0. Moreover, $\mathrm{res}_{ME_2/E_2}\alpha = \mathrm{res}_{ME_2/E_2}\alpha'$. So, changing if necessary α to α' we can assume that $\mathrm{Supp}(\alpha) \subset \{p_0\}$, where $\mathrm{Supp}(\alpha) = \{p \in X : \partial_p(\alpha) \neq 0\}$. Consider two possible cases:

A) $\mathrm{Supp}(\alpha) = \emptyset$. Let $\widetilde{\alpha} \in K_2(E_2)$ be a preimage of α under the norm residue map $K_2(E_2) \to {}_2\mathrm{Br}(E_2)$. Consider the commutative diagram

$$\begin{array}{ccc} K_2(E_2) & \longrightarrow & {}_2\mathrm{Br}\, E_2 \\ {\scriptstyle \partial}\downarrow & & {\scriptstyle \partial}\downarrow \\ \coprod_p E_1(p)^* & \longrightarrow & \coprod_p E_1(p)^*/E_1(p)^{*2} \end{array}$$

We need the following:

Lemma 8. *(See [Su].) Suppose k is a field, X is a nonsplit conic over k, $s \in K_2(k(X))$ is such that $\partial_p(s) \in k(p)^{*2}$ for any $p \in X$. Let $\partial_p(s) = t_p^2$, and put $t = \prod_{p \in X} \mathrm{N}_{k(p)/k}(t_p)$. Then t is independent of the choice of t_p, $t \in \{-1, 1\}$, and $t = 1$ if and only if $s \in 2K_2(k(X)) + \mathrm{res}_{K(X)/k} K_2(k)$.*

For any $p \in X$ we have $\partial_p(\widetilde{\alpha}) = x_p^2$ for some $x_p \in E_1(p)^*$. Since $Q_{ME_1} \neq 0$, and $\alpha_{ME_2} \in \operatorname{res}_{ME_2/ME_1} {}_2\mathrm{Br}\, ME_1$, Lemma 8 implies that $\prod_p N_{E_1(p)/E_1} x_p = 1$. This means that $\alpha = \gamma_{E_2}$ for some $\gamma \in {}_2\mathrm{Br}\, E_1$. So $(\beta - \gamma_{ME_1})_{ME_2} = 0$, which means that β equals either γ_{ME_1}, or $(\gamma + Q)_{ME_1}$. In both cases the image of β in $h_3(ME_1/E_1)$ is zero.

B) $\operatorname{Supp}(\alpha) = \{p_0\}$. We will show that this case is in fact impossible. Obviously, $\partial_{p_0}(\alpha) = s$ for some $1 \neq s \in G$. Consider the commutative diagram

$$\begin{array}{ccc} {}_2\mathrm{Br}\, E_1(\sqrt{s}) & \xrightarrow{\mathrm{res}} & {}_2\mathrm{Br}\, E_2(\sqrt{s}) \\ {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow \\ {}_2\mathrm{Br}\, E_1 & \xrightarrow{\mathrm{res}} & {}_2\mathrm{Br}\, E_2 \end{array}$$

The argument in case *A*) shows that $\alpha_{E_2(\sqrt{s})} = \gamma_{E_2(\sqrt{s})}$ for some $\gamma \in {}_2\mathrm{Br}\, E_1(\sqrt{s})$. Therefore,

$$\operatorname{res}_{E_2/E_1} \circ \mathrm{N}_{E_1(\sqrt{s})/E_1} \gamma = \mathrm{N}_{E_2(\sqrt{s})/E_2} \circ \operatorname{res}_{E_2(\sqrt{s})/E_1(\sqrt{s})} \gamma = \mathrm{N}_{E_2(\sqrt{s})/E_2} \circ \operatorname{res}_{E_2(\sqrt{s})/E_2} \alpha = 0.$$

This implies that $\mathrm{N}_{E_1(\sqrt{s})/E_1} \gamma$ equals either 0, or Q. If $\mathrm{N}_{E_1(\sqrt{s})/E_1} \gamma = Q$, then $Q \cup (s) = 0$, a contradiction to the hypothesis. So $\mathrm{N}_{E_1(\sqrt{s})/E_1} \gamma = 0$, i.e. $\gamma = \delta_{E_1(\sqrt{s})}$ for some $\delta \in {}_2\mathrm{Br}\, E_1$. Taking into account that $\alpha_{E_2(\sqrt{s})} = \gamma_{E_2(\sqrt{s})}$, we get

$$\alpha - \delta_{E_2} = (s, f)$$

for some $f \in E_2^*$. Since $\partial_{p_0}(\alpha) \neq 1$, there exist $p_1 \in X$ of degree not divisible by 4, $p_1 \neq p_0$, odd integers k_0, k_1, and a divisor $\mathfrak{a}$ such that $\operatorname{div}(f) = k_0 p_0 + k_1 p_1 + \mathfrak{a}$ and $p_0, p_1 \notin \operatorname{Supp} \mathfrak{a}$. Since

$$s = \partial_{p_1}(s, f) = \partial_{p_1}(\alpha) = 1,$$

we get $\sqrt{s} \in E_1(p_1)$. Moreover, $[E_1(p_1) : E_1(\sqrt{s})]$ is odd, because $[E_1(p_1) : E_1]$ is not divisible by 4. Since $Q_{E_1(p_1)} = 0$, we conclude that $Q_{E_1(\sqrt{s})} = 0$, hence $Q_{ME_1} = 0$, a contradiction.

4) Suppose $\varphi \simeq \langle a_1, a_2, a_3, a_4 \rangle$. Let $E = E_1(t_1, t_2, t_3)$, where t_1, t_2, t_3 are indeterminates. We have

$$\begin{aligned} \varphi_E &\simeq \big\langle a_1t_1^2 + a_2t_2^2, a_1a_2\big(a_1t_1^2 + a_2t_2^2\big), a_3, a_4 \big\rangle \\ &\simeq \big\langle a_1t_1^2 + a_2t_2^2 + a_3t_3^2, a_3\big(a_1t_1^2 + a_2t_2^2\big)\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2\big), a_1a_2\big(a_1t_1^2 + a_2t_2^2\big), a_4 \big\rangle \\ &\simeq \big\langle a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4, a_4\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2\big)\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4\big), \\ &\quad a_3\big(a_1t_1^2 + a_2t_2^2\big)\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2\big), a_1a_2\big(a_1t_1^2 + a_2t_2^2\big)\big\rangle. \end{aligned}$$

It follows that

$$\varphi_E \simeq \big\langle a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4 \big\rangle \perp \varphi',$$

for some 3-dimensional form φ' over E, and

$$C_0(\varphi') \simeq \big(-a_1a_2a_3\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2\big), -a_3a_4\big(a_1t_1^2 + a_2t_2^2\big)\big(a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4\big)\big).$$

Hence $\partial_{a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4} C_0(\varphi') = \operatorname{disc}(\varphi)$, which implies

$$\partial_{a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4}\big(C_0(\varphi') \cup (s)\big) = \big(\operatorname{disc}(\varphi), s\big).$$

Since $\operatorname{disc}\varphi \neq 1$, and $(\operatorname{disc}\varphi, s) \neq 0$, it follows that $(\operatorname{disc}(\varphi), s)_{E_1(\varphi)} \neq 0$. Hence $C_0(\varphi') \cup (s) \neq 0$. Since the extensions $E(\varphi')(\varphi)/E(\varphi')$ and E/E_1 are purely transcendental, the result follows from parts 2) and 3).

5) Since the discriminant of the generic 4-dimensional subform of φ is equal to $a_1t_1^2 + a_2t_2^2 + a_3t_3^2 + a_4t_4^2 + a_5$, the result follows from part 4). $\square$

Now we are going to apply the Merkurjev construction of the field with a prescribed even u-invariant (see [M2], Theorem 3). Starting from any field F construct an infinite tower of fields $F = F_0 \subset F_1 \subset \cdots$ in the following way: if i is even, then F_{i+1} is a free composite of all function fields $F_i(\varphi)$, φ ranging over all 5-dimensional forms over F_i. If i is odd, then F_{i+1} is a maximal odd degree extension of F_i. Put $\widehat{F} = \bigcup_i F_i$. It is easy to check that the u-invariant of $\widehat{F}$ is not more than 4, and $\widehat{F}$ has not any proper odd degree extension. In particular, $\operatorname{cd}_2 \widehat{F} \leqslant 2$. In view of Proposition 7 we have $h_3(M/F) \hookrightarrow h_3(M\widehat{F}/\widehat{F})$. Therefore, using the remark after the proof of Corollary 5, we can give an example of a field $\widehat{F}$ such that the group $h_3(M\widehat{F}/\widehat{F})$ is infinite, and $\operatorname{cd}_2 \widehat{F} \leqslant 2$.

Remark. If any 4-dimensional form with nontrivial discriminant over a field F is isotropic, and M/F is an arbitrary multiquadratic extension, then $h_3(M/F) = 0$ (see [ELW], Corollary 4.7). Thus, part 4) of Proposition 7 does not generalize to any 4-dimensional form with nontrivial discriminant.

2. The group $h_1(M/F)$ for triquadratic extensions

Examples of triquadratic extensions (multiquadratic extensions of degree 8) M/F with nontrivial $h_1(M/F)$ are known (see [STW], Theorem 5.1, Remark 5.4). However, in all these examples $\operatorname{char} F = 0$. In this section we get rid of this restriction. First we recall the definition of property CV for a quadratic extension L/F.

By definition, a quadratic extension L/F has property CV (common value), if for any two binary quadratic forms q_1, q_2 over F the existence of a common value of the forms q_{1L}, q_{2L} implies the existence of a common value of the forms q_{1L}, q_{2L}, lying in F^*. Examples of quadratic extensions of characteristic 0 not satisfying this property has been given in [STW], Remark 5.4. In the same paper it was established that for a triquadratic extension $F(\sqrt{a}, \sqrt{b}, \sqrt{c})/F$

$$h_1\big(F(\sqrt{a}, \sqrt{b}, \sqrt{c})/F\big) \simeq \frac{F^* \cap N_{F(\sqrt{c})}(a) N_{F(\sqrt{c})}(b)}{(F^* \cap N_{F(\sqrt{c})}(a))(F^* \cap N_{F(\sqrt{c})}(b))},$$

where we denote by $N_K(u)$ the norm group $N_{K(\sqrt{u})/K} K(\sqrt{u})^*$ for a field extension $K(\sqrt{u})/K$. Moreover, if forms $\alpha \simeq \langle 1, -a\rangle$ and $\beta \simeq z\langle 1, -b\rangle$ give a counterexample to property CV for an extension $F(\sqrt{c})/F$, then $(a, z)_{F(\sqrt{ab}, \sqrt{c})} = 0$, and $a \otimes z + ab \otimes u + c \otimes v$ is a nontrivial element in $h_1(F(\sqrt{a}, \sqrt{b}, \sqrt{c})/F)$ for any u, v such that $(a, z) + (ab, u) + (c, v) = 0$.

On the other hand, suppose k is a field, $d, c_1, c_2 \in k^*$, $d \notin k^{*2}$ such that $c_1^2 - 4d$ and $c_2^2 - 4d$ are nontrivial distinct elements in k^*/k^{*2}, x is an indeterminate, $F = k(x)$, $L = F(\sqrt{x^2 - 4d})$. Set

$$q_1 \simeq (x - c_1)\langle\langle c_1^2 - 4d\rangle\rangle, \qquad q_2 \simeq (x - c_2)\langle\langle c_2^2 - 4d\rangle\rangle.$$

It has been shown in [S2], Proposition 1, that the extension L/F does not satisfy property CV, and the forms q_1, q_2 provide a corresponding counterexample. Here we have $a = c_1^2 - 4d$, $b = c_2^2 - 4d$, $c = x^2 - 4d$, $z = (x - c_1)(x - c_2)$. It is easy to check computing the residues that

$$\begin{aligned}&\big(a, (x - c_1)(x - c_2)\big) + \big(ab, (c_1 + c_2)(x - c_2)\big((c_1 + c_2)x - c_1c_2 - 4d\big)\big)\\ &\quad + \big(x^2 - 4d, -(x - c_1)(x - c_2)\big((c_1 + c_2)x - c_1c_2 - 4d\big)\big) = 0.\end{aligned}$$

Therefore, in view of the discussion above we get the following:

Proposition 9. *The element*

$$\begin{aligned}&\big(c_1^2-4d\big)\otimes(c_1+c_2)(x-c_1)\big((c_1+c_2)x-c_1c_2-4d\big)\\&\quad+\big(c_2^2-4d\big)\otimes(c_1+c_2)(x-c_2)\big((c_1+c_2)x-c_1c_2-4d\big)\\&\quad+\big(x^2-4d\big)\otimes-(x-c_1)(x-c_2)\big((c_1+c_2)x-c_1c_2-4d\big)\end{aligned}$$

is nontrivial in the homology group $h_1(F(\sqrt{c_1^2-4d},\sqrt{c_2^2-4d},\sqrt{x^2-4d})/F)$.

For an arbitrary field F and elements $a,b,c\in F^*$ the group $h_1(F(\sqrt{a},\sqrt{b},\sqrt{c})/F)$ can be written in a form symmetric with respect to a,b,c, namely,

$$h_1\big(F(\sqrt{a},\sqrt{b},\sqrt{c})/F\big)\simeq\frac{N_F(a)\cap N_F(b)\cap N_F(c)}{F^{*2}N_{F(\sqrt{a},\sqrt{b},\sqrt{c})/F}F(\sqrt{a},\sqrt{b},\sqrt{c})^*}$$

(see [G], Proposition 3). To do this let $u\in F^*, u=\alpha\beta$, where $\alpha\in N_{F(\sqrt{c})}(a)$, $\beta\in N_{F(\sqrt{c})}(b)$. Then the rule $u\to N_{F(\sqrt{c})/F}(\alpha)$ determines a well-defined isomorphism

$$\frac{F^*\cap N_{F(\sqrt{c})}(a)N_{F(\sqrt{c})}(b)}{(F^*\cap N_{F(\sqrt{c})}(a))(F^*\cap N_{F(\sqrt{c})}(b))}\simeq\frac{N_F(a)\cap N_F(b)\cap N_F(c)}{F^{*2}N_{F(\sqrt{a},\sqrt{b},\sqrt{c})/F}F(\sqrt{a},\sqrt{b},\sqrt{c})^*}. \qquad (**)$$

It remains to notice that the left-hand group is isomorphic to $h_1(F(\sqrt{a},\sqrt{b},\sqrt{c})/F)$.

In our example where $F=k(x)$ it is easy to check, computing the residues, that

$$\big(c_1^2-4d,2(x-c_1)\big(x+\sqrt{x^2-4d}\big)\big)=\big(c_2^2-4d,2(x-c_2)\big(x+\sqrt{x^2-4d}\big)\big)=0$$

(recall that $x+\sqrt{x^2-4d}=2t$), which means that

$$\begin{aligned}2(x-c_1)\big(x+\sqrt{x^2-4d}\big)&\in N_{F(\sqrt{x^2-4d})}\big(c_1^2-4d\big),\\2(x-c_2)\big(x+\sqrt{x^2-4d}\big)&\in N_{F(\sqrt{x^2-4d})}\big(c_2^2-4d\big).\end{aligned}$$

Under isomorphism $(**)$ the image of the element

$$z=(x-c_1)(x-c_2)\in N_{F(\sqrt{x^2-4d})}\big(c_1^2-4d\big)N_{F(\sqrt{x^2-4d})}\big(c_2^2-4d\big)$$

is $N_{F(\sqrt{c})/F}2(x-c_1)(x+\sqrt{x^2-4d})=d\in F^*/F^{*2}$. Summarizing, we get the following:

Corollary 10. *Let k be a field, $a_1,a_2\in k^*/k^{*2}$ distinct nontrivial elements, $d\in k^*\setminus k^{*2}$, and $(a_1,d)=(a_2,d)=0$. Put $F=k(x)$, $M=F(\sqrt{a_1},\sqrt{a_2},\sqrt{x^2-4d})$. Then $d\notin F^{*2}N_{M/F}M^*$.*

Proof. Since $(a_i,d)=0$, we get $a_i=c_i^2e_i^2-4de_i^2$ for some $c_i,e_i\in F$. Now the corollary follows from the argument above. □

Corollary 11. *In notation of Corollary* 10 *consider the division algebra* $A \simeq (a_1, X_1) \otimes (a_2, X_2) \otimes (x^2 - 4d, X_3) \in {}_2\mathrm{Br}\, E$, *where* $E = F((X_1))((X_2))((X_3))$. *Then* $A \cup (d) = 0 \in H^3(E, \mathbb{Z}/2\mathbb{Z})$, *but* $d \notin E^{*2}\, \mathrm{Nrd}(A^*)$.

Proof. It is obvious that $A \cup (d) = 0$. Since $d \notin F^{*2} N_{M/F} M^*$, we have $d \notin E^{*2}\, \mathrm{Nrd}(A^*)$ (see [KLST, p. 283]). □

Corollary 12. *In notation of Corollary* 11 *consider the form*

$$\Phi \simeq \langle\langle a_1, X_1\rangle\rangle \perp -\langle\langle a_2, X_2\rangle\rangle \perp a_1\langle\langle x^2 - 4d, X_3\rangle\rangle.$$

Then the E*-variety* $PSO(\Phi_{an})$ *is not stably rational, i.e.* $PSO(\Phi_{an}) \times \mathbb{A}_E^m \not\simeq \mathbb{A}_E^n$ *for any* m, n.

Proof. The result follows from [G, p. 745] (the proof in [G] is given only for the zero characteristic case, but inspection shows that this restriction is not in fact necessary). □

Starting from the extension in Corollary 10 it is easy to produce a triquadratic extension $\widehat{M}/\widehat{F}$ such that the group $h_1(\widehat{M}/\widehat{F})$ is infinite. Put $a_3 = x^2 - 4d$. Let X_n be the affine F-variety determined by the equations

$$X_{i1}^2 - a_1 Y_{i1}^2 - T_i = X_{i2}^2 - a_2 Y_{i2}^2 - T_i = X_{i3}^2 - a_3 Y_{i3}^2 - T_i = 0 \quad (1 \leqslant i \leqslant n).$$

Put $F_n = F(X_n)$, $M_n = MF_n$. Obviously, $(a_1, T_i) = (a_2, T_i) = (a_3, T_i) = 0$ for any $1 \leqslant i \leqslant n$.

Proposition 13.

1) *The elements* T_i *are pairwise distinct in the group*

$$\frac{N_{F_n}(a_1) \cap N_{F_n}(a_2) \cap N_{F_n}(a_3)}{F_n^{*2} N_{M_n/F_n} {M_n}^*} \simeq h_1(M_n/F_n).$$

2) *Put* $\widehat{F} = \bigcup_{i=1}^{\infty} F_i$, $\widehat{M} = M\widehat{F}$. *Then the group* $h_1(\widehat{M}/\widehat{F})$ *is infinite.*

Proof. 1) We should prove that all the elements $T_i T_j$ are nontrivial in this group. The variety X_n has a nonsingular rational point p with $T_i = d$, $T_j = 1$. It determines a discrete F-valuation, and the extension M_n/F_n is unramified with respect to this valuation. Choose a local parameter $\pi \in F_n^*$. Let s be the specialization map $M_n^* \to M^*$ with respect to π. If $z = \pi^n u \in M_n^*$, and $v(u) = 0$, then by definition $s(z) = \bar{u} \in M^*$. Commutativity of the diagram

$$\begin{array}{ccc} M_n^* & \xrightarrow{s} & M^* \\ {\scriptstyle N}\big\downarrow & & {\scriptstyle N}\big\downarrow \\ F_n^* & \xrightarrow{s} & F^* \end{array}$$

shows that triviality of $T_i T_j$ in $h_1(M_n/F_n)$ would imply triviality of $d = s(T_i T_j)$ in $h_1(M/F)$.

2) Obvious, in view of part 1). □

Remark. If the field F is of finite type over the prime field, then the group $\frac{N_F(a_1) \cap \cdots \cap N_F(a_n)}{F^{*2} N_{M/F} M^*}$ is finite for any multiquadratic extension $M = F(\sqrt{a_1}, \ldots, \sqrt{a_n})/F$ (see [G], Remark 3). In particular, the group $h_1(M/F)$ is finite if M/F is triquadratic.

Open questions.

1) Let M/F be any multiquadratic extension. Does there exist an extension K/F linearly disjoint with M/F such that $h_1(MK/K) \neq 0$ (resp. $h_3(MK/K) \neq 0$)?
2) Suppose F is of finite type over the prime field. Is the group $h_1(M/F)$ (resp. $h_3(M/F)$) finite?

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